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THE NUMERICAL COMPUTATION OF THREE-DIMENSIONAL TURBULENT BOUNDA--ETC(U)
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SUMMARY

The governing equations for compressible three-dimensional turbulent boundary layers in a general coordinate system are given. Methods for the solution of these equations and their integral counterparts are described and compared. The available finite difference techniques are discussed in detail.

Invited lecture presented at the IUTAM Symposium on Three-Dimensional Turbulent Boundary Layers, Technische Universität, Berlin, [illegible]-31 March 198[illegible]

1 INTRODUCTION

I would like to begin by saying what an honour and a pleasure it is for me to give this review lecture. To some extent my pleasure is enhanced by a sense of déja vu as it is almost exactly ten years since I spoke on a similar topic at this very Institute. The occasion was then EUROMECH 33 [1] on three-dimensional turbulent boundary layers. At that meeting I described the integral calculation method[2] I had developed for compressible turbulent boundary layers and laid particular stress upon the generality of the coordinate system I had adopted. At that time the method was probably the most advanced (at least in its range of applicability) available and will serve as a useful datum against which to measure progress during the past decade. Of course, as I hope to show, real progress has been made. Other integral methods have adopted the general coordinate system and have used more sophisticated assumptions about the velocity profiles and entrainment. The major advance has been, however, in the development of the finite-difference methods and I hope to spend some time describing these in detail. As a guard against complacency let me state here however, that progress has not been achieved in the types of turbulence modelling we use in these methods. In 1972 Wesseling and Lindhout[3] described their finite-difference method which although confined to incompressible flow and Cartesian coordinates used Bradshaw's[4] vector equation for the turbulent shear stress. Ten years later almost all the finite-difference methods appear to have reverted to the more primitive eddy-viscosity concept of turbulence modelling.

EUROMECH 33 also marked the birth of another significant development in this field, the organisation of a series of meetings to compare results from the various calculations methods with each other and where possible against experiment. The first of these meetings, EUROMECH 60, became known as the Trondheim Trials[5] and prompted the EUROVISC Committee of national representatives to form a working party which has since organised two further meetings, at Stockholm[6] in 1978 and Amsterdam[7] in 1979. Finally, as you will all know a third meeting will immediately follow this Symposium so that any predictions I may be foolish enough to make concerning the capabilities of the various calculation methods will rapidly be put to the test and almost certainly shown to be false!

As with any review paper little of what follows is original. In particular I have drawn heavily upon the works of J.P.F. Lindhout[7] of the NLR and my colleague at the RAE, Caroline Boyd[8]. To them and also to all those whose names appear in the list of references I offer my sincere thanks.

2 THE THREE-DIMENSIONAL BOUNDARY LAYER EQUATIONS[9-11]

For general applicability most three-dimensional boundary-layer calculation methods use a non-orthogonal curvilinear coordinate system; x, y in the surface, z normal to the surface with an angle λ (x,y) between the x and y directions. In this coordinate system the continuity, momentum and energy equations may be written as.

Continuity:

$$\frac{\partial}{\partial x}(\rho u h_2 \sin\lambda) + \frac{\partial}{\partial y}(\rho v h_1 \sin\lambda) + \frac{\partial}{\partial z}(\overline{\rho w} h_1 h_2 \sin\lambda) = 0 \qquad (1)$$

x momentum:

$$\frac{\rho u}{h_1}\frac{\partial u}{\partial x} + \frac{\rho v}{h_2}\frac{\partial u}{\partial y} + \overline{\rho w}\frac{\partial u}{\partial z} - \rho \cot\lambda\, k_1 u^2 + \rho \csc\lambda\, k_2 v^2 + \rho k_{12} uv$$

$$= -\csc^2\frac{\lambda}{h_1}\frac{\partial p}{\partial x} + \cot\lambda\,\frac{\csc\lambda}{h_2}\frac{\partial p}{\partial y} + \frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z} - \overline{\rho u'w'}\right) , \qquad (2)$$

y momentum:

$$\frac{\rho u}{h_1}\frac{\partial v}{\partial x} + \frac{\rho v}{h_2}\frac{\partial v}{\partial y} + \overline{\rho w}\frac{\partial v}{\partial z} - \rho \cot\lambda\, k_2 v^2 + \rho \csc\lambda\, k_1 u^2 + \rho k_{21} uv$$

$$= \cot\lambda\,\frac{\csc\lambda}{h_1}\frac{\partial p}{\partial x} - \frac{\csc^2\lambda}{h_2}\frac{\partial p}{\partial y} + \frac{\partial}{\partial z}\left(\mu\frac{\partial v}{\partial z} - \overline{\rho v'w'}\right) , \qquad (3)$$

energy equation:

$$\frac{\rho u}{h_1}\frac{\partial H}{\partial x} + \frac{\rho v}{h_2}\frac{\partial H}{\partial y} + \overline{\rho w}\frac{\partial H}{\partial z} = \frac{\partial}{\partial z}\left[\frac{\mu}{P_r}\frac{\partial H}{\partial z} + \mu\left(1 - \frac{1}{P_r}\right)\frac{\partial}{\partial z}\left(\frac{U^2}{2}\right) - \overline{\rho w'H'}\right] . \qquad (4)$$

Here

$$\overline{\rho w} = \rho w + \overline{\rho' w'} .$$

An element of length on the surface ds is given by

$$ds^2 = h_1^2 dx^2 + h_2^2 dy^2 + 2h_1 h_2 \cos\lambda\, dxdy \qquad (5)$$

and the total velocity U is given by

$$U^2 = u^2 + v^2 + 2uv\cos\lambda \qquad (6)$$

where the metric coefficients h_1 and h_2 are to the boundary layer approximation functions of x and y but not z, *ie*

$$h_1 = h_1(x,y) , \qquad h_2 = h_2(x,y) .$$

The boundary conditions are

$$\left.\begin{array}{llll} z = 0 ; & u = v = 0 , & w = w_w , & \left(\frac{\partial H}{\partial y}\right)_w = \text{given} \\ z = \delta ; & u = u_e(x,y) , & v = v_e(x,y) , & H = H_e(x,y) . \end{array}\right\} \qquad (7)$$

The curvature parameters k_1, k_2, k_{12} and k_{21} are given by

$$k_1 = \frac{1}{h_1 h_2 \sin\lambda}\left[\frac{\partial}{\partial x}(h_2 \cos\lambda) - \frac{\partial h_1}{\partial y}\right] \tag{8}$$

$$k_2 = \frac{1}{h_1 h_2 \sin\lambda}\left[\frac{\partial}{\partial y}(h_1 \cos\lambda) - \frac{\partial h_2}{\partial x}\right] \tag{9}$$

$$k_{12} = \frac{1}{\sin\lambda}\left[-\left(k_1 + \frac{1}{h_1}\frac{\partial\lambda}{\partial x}\right) + \cos\lambda\left(k_2 + \frac{1}{h_2}\frac{\partial\lambda}{\partial y}\right)\right] \tag{10}$$

$$k_{21} = \frac{1}{\sin\lambda}\left[-\left(k_2 + \frac{1}{h_2}\frac{\partial\lambda}{\partial y}\right) + \cos\lambda\left(k_1 + \frac{1}{h_1}\frac{\partial\lambda}{\partial x}\right)\right] . \tag{11}$$

For an orthogonal system, $\lambda = 90^o$, $k_1 = -k_{12}$, $k_2 = -k_{21}$.

Rather than solve the energy equation (4), for adiabatic flow the density at each point in the boundary layer is frequently assumed to be approximated by the Crocco relation

$$\frac{\rho e}{\rho} = 1 + \frac{\gamma - 1}{2} r M_e^2 \left(1 - \frac{U^2}{U_e^2}\right) \tag{12}$$

where r , the recovery factor is taken as 0.84 or 0.85 for laminar flows and 0.89 for turbulent flow.

At the boundary-layer edge equations (2) and (3) reduce to

$$\rho e\left(\frac{u_e}{h_1}\frac{\partial u_e}{\partial x} + \frac{v_e}{h_2}\frac{\partial u_e}{\partial y} - \cot\lambda\, k_1 u_e^2 + \csc\lambda\, k_2 v_e^2 + k_{12} u_e v_e\right) = -\frac{\csc^2\lambda}{h_1}\frac{\partial p}{\partial x} + \frac{\cot\lambda \csc\lambda}{h_2}\frac{\partial p}{\partial y}$$

...... (13)

and

$$\rho e\left(\frac{u_e}{h_1}\frac{\partial v_e}{\partial x} + \frac{v_e}{h_2}\frac{\partial v_e}{\partial y} - \cot\lambda\, k_2 v_e^2 + \csc^2\lambda\, k_1 u_e^2 + k_{21} u_e v_e\right) = \frac{\cot\lambda \csc\lambda}{h_1}\frac{\partial p}{\partial x} - \frac{\csc^2\lambda}{h_2}\frac{\partial p}{\partial y} .$$

...... (14)

The two Euler equations, (13) and (14), may be used to eliminate p from equations (2) and (3), or for a given pressure distribution p (x,y) may be solved for u_e and v_e subject to appropriate initial and boundary conditions.

3 INTEGRAL METHODS

The earliest and simplest attempts to find solutions to the general equations (1) to (3) involved the use of integral methods in which the equations are integrated term by term across the boundary layer from the wall to the outer edge. These integral equations were first derived by Myring[12]. To proceed further it is then necessary to assume some

form for the velocity profiles in the boundary layer. All integral methods make use of streamwise and crosswise profiles. That is profiles in the directions of the external streamlines (s) and their orthogonal trajectories (n) respectively. It was therefore necessary for Myring to derive the transformation relationships between the integral thicknesses in the x, y coordinate system and the corresponding thicknesses in the s, n coordinate system. The equations may however be derived rather more directly by starting from the integral equations in streamline coordinates[13] and transforming the independent variables s, n to x, y as shown below. This derivation will be useful when we come to discuss inverse methods.

The integral equations in streamline coordinates are:

s momentum:

$$\frac{\partial\theta_{11}}{\partial s} + \frac{\partial\theta_{12}}{\partial n} + \theta_{11}\left(\frac{H+2-M_e^2}{U_e}\frac{\partial U_e}{\partial s} - k_s\right) + \theta_{12}\left(\frac{2-M_e^2}{U_e}\frac{\partial U_e}{\partial n} - 2k_n\right) + \delta_2\left(\frac{1}{U_e}\frac{\partial U_e}{\partial n} - k_n\right) + k_2\theta_{22} = \frac{C_{fs}}{2} \tag{15}$$

n momentum:

$$\frac{\partial\theta_{21}}{\partial s} + \frac{\partial\theta_{22}}{\partial n} + \theta_{21}\left(\frac{2-M_e^2}{U_e}\frac{\partial U_e}{\partial s} - 2k_s\right) + k_n\theta_{11}(H+1) + \theta_{22}\left(\frac{2-M_e^2}{U_e}\frac{\partial U_e}{\partial n} - k_n\right) = \frac{C_{fn}}{2} \tag{16}$$

continuity:

$$\frac{\partial(\delta-\delta_1)}{ds} - \frac{\partial\delta_2}{\partial n} + (\delta-\delta_1)\left[\frac{1-M^2}{U_e}\frac{\partial U_e}{\partial s} - k_s\right] - \delta_2\left(\frac{1-M_e^2}{U_e}\frac{\partial U_e}{\partial n} - k_n\right) = C_E \tag{17}$$

where C_E is the entrainment coefficient.

These equations are transformed to the general x, y non-orthogonal coordinate system by using the operators

$$\frac{\partial}{\partial s} = \frac{\sin(\lambda-\alpha)}{\sin\lambda}\frac{1}{h_1}\frac{\partial}{\partial x} + \frac{\sin\alpha}{\sin\lambda}\frac{1}{h_2}\frac{\partial}{\partial y} \tag{18}$$

$$\frac{\partial}{\partial n} = -\frac{\cos(\lambda-\alpha)}{\sin\lambda}\frac{1}{h_1}\frac{\partial}{\partial x} + \frac{\cos\alpha}{\sin\lambda}\frac{1}{h_2}\frac{\partial}{\partial y} \tag{19}$$

where λ is the angle between the x and y directions and α the angle between the x and s directions so that

$$\sin\alpha = \frac{V_e}{U_e}\sin\theta \quad . \tag{20}$$

The curvature terms k_s and k_n are derived as follows. For k_s we equate the continuity equation (1) for the external flow in the x, y coordinate system to its counterpart in the s, n coordinate system. The result is

$$k_s = -\frac{1}{h_1 h_2 \sin\theta}\left[\frac{\partial}{\partial x}\left[h_2 \sin(\theta-\alpha)\right] + \frac{\partial}{\partial y}\left(h_1 \sin\alpha\right)\right] \quad . \tag{21}$$

For k_n we equate the equation for the component of vorticity of the external flow in the direction normal to the surface[14].

$$\zeta = \frac{1}{h_1 h_2 \sin\theta}\left[\frac{\partial}{\partial x}\left(h_2 U_e \cos(\theta-\alpha)\right) - \frac{\partial}{\partial y}\left(h_1 U_e \cos\alpha\right)\right] \tag{22}$$

to its counterpart in the s, n coordinate system

$$\zeta = k_n U_e - \frac{\partial U_e}{\partial n} \tag{23}$$

with the result that

$$k_n = \frac{1}{h_1 h_2 \sin\theta}\left[\frac{\partial}{\partial x}\left(h_2 \cos(\theta-\alpha)\right) - \frac{\partial}{\partial y}\left(h_1 \cos\alpha\right)\right] \quad . \tag{24}$$

We note from equation (23) that if the external flow is irrotational then $k_n = (1/U_e)(\partial U_e/\partial n)$ and some simplification of equations (15) to (17) results.

Up to this point no approximations have been made but to proceed further it is necessary to assume forms for the velocity profiles in order to derive relationships between the unknowns so that they may be reduced in number to three. The various assumptions which have been made are given below.

Smith[2] assumed power law profiles for the streamwise flow and Mager[15] profiles for the crossflow. Streamwise skin friction and the entrainment coefficient were derived from auxiliary relationships. Later the method was modified so that the entrainment coefficient was calculated by an extension of the lag-entrainment method of Green, *et al*[16].

Stock[17] modified Smith's method so that the streamwise profile was represented by Coles law of the wall plus wake. This removed the need for an empirical skin friction relationship. In addition the entrainment was calculated from Horton's lag entrainment relationship.

Cross[18] uses a three-dimensional form of the law of the wall and wake together with Thompson's entrainment relationship.

Cousteix[19,20] and Aupoix use velocity profiles derived from an analysis of similarity solutions.

The methods are listed above in order of increasing complexity. They all use very similar numerical methods to solve the equations and it is interesting to note that this increase in complexity is reflected in increased computing times as listed by Lindhout[7] for the Amsterdam Workshop test case. The times are shown in Table 1 in terms of effort measured in Mega Floating Point Operations (MFLOP) per surface point.

Table 1

Method	Effort per point (MFLOP)
Smith	0.007
Stock	0.013
Cross	0.018
Cousteix and Aupoix	0.033

These figures must be interpreted with caution. The methods have usually been written with clarity and ease of alteration rather than speed as the prime objective and in addition the numerical method of Cousteix and Aupoix is more complicated than those by the others. The important point to note is that all the methods are so fast that realistic configurations can be calculated in very acceptable run times and even the slowest integral method remains an order of magnitude faster than a finite difference method.

The assumption of velocity profiles and where necessary additional empirical relationships reduces the number of unknowns to three (A, B, C say) and, typically, three equations of the form

$$\left.\begin{aligned}
\frac{\partial A}{\partial x} &= g1\left(A,\ B,\ C,\ \frac{\partial A}{\partial y},\ \frac{\partial B}{\partial y},\ \frac{\partial C}{\partial y},\ U_e,\ \alpha,\ \frac{\partial U_e}{\partial x},\ \frac{\partial U}{\partial y},\ \frac{\partial \alpha}{\partial x},\ \frac{\partial \alpha}{\partial y},\right.\\
&\qquad\left. h_1,\ h_2,\ \lambda,\ \frac{\partial h_1}{\partial x},\ \frac{\partial h_1}{\partial y},\ \frac{\partial h_2}{\partial x},\ \frac{\partial h_2}{\partial y},\ \frac{\partial \lambda}{\partial x},\ \frac{\partial \lambda}{\partial y}\right)\\
\frac{\partial B}{\partial x} &= g2\ (A,\ \text{etc})\\
\frac{\partial C}{\partial x} &= g3\ (A,\ \text{etc})
\end{aligned}\right\} \qquad (25)$$

where all the boundary layer parameters are now known functions of A, B, C and U_e.

The equations are hyperbolic[12] with three real characteristics and it is necessary to take account of these characteristic directions when evaluating the y derivatives as shown in Fig 1. To a good approximation the upper and lower bounds of the characteristic directions with respect to the x axis are given by α and $\alpha + \beta$ respectively where β is the direction of the wall streamline with respect to s. With the y derivatives evaluated in this manner the solution is advanced in the x direction by a simple

explicit method, Smith[2] for example uses a two step Euler method whilst Cousteix and Aupoix[19,20] use a fourth order Runge Kutta method. The explicit nature of the method restricts the forward step size to that allowed by the Courant Friedrichs Levy condition but as noted above run times are short and no attempt has been made to remove this restriction by developing an implicit method.

4 DISPLACEMENT THICKNESS AND EQUIVALENT SOURCE DISTRIBUTION

For calculations involving viscous-inviscid interactions the effect of the boundary layer upon the external flow may be represented as a change in boundary condition for the inviscid calculation from one of zero velocity normal to the surface to one in which the velocity component normal to the surface, m , is given by

$$m = \frac{1}{\rho_e h_1 h_2 \sin\lambda}\left[\frac{\partial}{\partial x}\left(\rho_e U_e h_2\left(\delta_1 \sin(\lambda-\alpha) - \delta_2 \cos(\lambda-\alpha)\right)\right) + \frac{\partial}{\partial y}\left[\rho_e U_e h_1(\delta_1 \sin\alpha + \delta_2 \cos\alpha)\right]\right] \quad (26)$$

m is often called the transpiration velocity or equivalent source strength.

Alternatively the effect of the boundary layer may be represented as a displacement of the surface through a distance δ^* normal to the surface where δ^* is given by

$$\frac{\partial}{\partial x}(\rho_e U_e h_2 \sin\lambda\delta^*) + \frac{\partial}{\partial y}(\rho_e v_e h_1 \sin\lambda\delta^*) = \rho_e h_1 h_2 \sin\lambda m \quad (27)$$

an additional partial differential equation which must be solved if this approach is adopted.

5 INVERSE INTEGRAL METHODS

For the calculation of separated flows it is necessary to invert the method so that the boundary layer development is specified and the velocity distribution is calculated. We can readily see that if two of the boundary layer thicknesses (B, C say) are specified as functions of x and y in equations (25) then the equations may be rearranged to the form

$$\left.\begin{aligned}\frac{\partial A}{\partial x} &= f1\left(A, U_e, \alpha, \frac{\partial A}{\partial y}, \frac{\partial U_e}{\partial y}, \frac{\partial \alpha}{\partial y}, B, C, \frac{\partial B}{\partial x}, \frac{\partial B}{\partial y}, \frac{\partial C}{\partial x}, \frac{\partial C}{\partial y}, h_1, h_2, \text{etc}\right)\\ \frac{\partial U_e}{\partial x} &= f2(A, \text{etc})\\ \frac{\partial \alpha}{\partial x} &= f3(A, \text{etc}).\end{aligned}\right\} \quad (28)$$

Solutions of this type for infinite yawed wings, in which $(\partial/\partial y) = 0$, have been published by Cousteix[21] and Stock[22]. In general, however, the inverse method will be part of an interactive calculation and only the source strength m or the displacement thickness δ^* will be available as input to the inverse method. An additional equation is

then required and this is given by equation (22). The inverse method is then a simultaneous solution of equations (25), (26) or (27) and (22). Whilst this is obviously possible in principle the characteristic directions may cause difficulties for any marching method and no working method of this type has yet been published.

6 FINITE-DIFFERENCE METHODS

The alternative to the integral approach is to approximate the terms of equations (1) to (4) by finite differences. If this is done in a consistent manner the solution of the difference equations will hopefully converge to the solution of the differential equations as the step sizes are reduced to zero. The earliest methods strongly resembled the numerical techniques adopted for integral methods in that they were explicit. That is to obtain the solution at the point x_i, y_j, z_k the solution is written in terms of the unknowns at x_i, y_j, z_k and known values in, the y, z plane at x_{i-1}. Upwind differencing is usually used in the y direction as shown in Fig 1. More recently the trend has been to implicit methods in which the solution is sought simultaneously along a complete z column at x_i, y_j. Once again upwind differencing is used in the y direction and central differences are used to evaluate the second derivative which occurs in the z direction. Typical of this type of method is that of Chang and Patel[25]. It will be seen from Fig 1 that no stability restrictions arise from the CFL condition for this approach but the method is restricted to flows in which v, the velocity in the y direction, is everywhere positive. If v is positive the methods of Lindhout[26] and Nash and Scruggs[27] use differencing techniques identical to those of Chang and Patel. Those used by Mclean[28] are similar but in addition Mclean uses three point backward differences in an attempt to preserve second order accuracy. Once v becomes negative it is necessary to evaluate the y derivative at the k_{i-1} station and then these methods are subject to the stability restriction that the characteristic (streamline) through x_i, y_j, z_k must intersect $x_{i-1}, j < y < j+1$. The zig-zag technique used by Kordulla[29], and originally formulated by Krause[30], is similar but here rather than switch operators as v changes sign the same second order operator is used throughout as shown in Fig 2. In addition second order accuracy in the x direction is achieved by centering the difference equations on M, $(x_{i-\frac{1}{2}}, y_j, z_k)$ as shown in Fig 2. Hence derivatives for constant Δy may be written

$$\left.\begin{aligned}
\left(\frac{\partial(\ )}{\partial x}\right)_M &= \frac{1}{\Delta x}\Big[(\ )_{i,j} - (\ )_{i-1,j}\Big]_K \\
\left(\frac{\partial(\ )}{\partial y}\right)_M &= \frac{1}{2\Delta y}\Big[(\ )_{i,j} - (\ )_{i,j-1} + (\ )_{i-1,j+1} - (\ )_{i-1,j}\Big]_K .
\end{aligned}\right\} \qquad (29)$$

If varying step sizes $\Delta y_1, \Delta y_2$ say are used then second order accuracy is only preserved if $\Delta y_2 - \Delta y_1$ is kept of order $\Delta y_1 \cdot \Delta y_2$. Similar restrictions apply to varying step sizes in the z direction. In order to overcome these restrictions and permit arbitrarily varying step sizes whilst preserving second order accuracy Cebeci[31] has used the box scheme. Here as shown in Fig 3 the equations are centred on $M(x_{i-\frac{1}{2}}, y_{j-\frac{1}{2}}, z_{k-\frac{1}{2}})$ and $(\ )_{i-\frac{1}{2},j-\frac{1}{2},k-\frac{1}{2}}$ is taken as the average of the eight corner values. Derivatives are written as

$$\left[\frac{\partial(\ )}{\partial x}\right]_M = \frac{1}{2\Delta x}\left[(\ )_{i,j} + (\ )_{i,j-1} - (\ )_{i-1,j} - (\ )_{i-1,j-1}\right]_{k-\frac{1}{2}}$$

$$\left[\frac{\partial(\ )}{\partial y}\right]_M = \frac{1}{2\Delta y}\left[(\ )_{i,j} + (\ )_{i-1,j} - (\ )_{i,j-1} - (\ )_{i-1,j-1}\right]_{k-\frac{1}{2}} \qquad (30)$$

$$\left[\frac{\partial(\ )}{\partial z}\right]_M = \frac{1}{4\Delta z}\Big[(\ )_{i,j,k} + (\ )_{i-1,j,k} + (\ )_{i,j-1,k} + (\ )_{i-1,j-1,k} - (\ )_{i,j,k-1} - (\ )_{i-1,j,k-1} - (\ )_{i,j-1,k-1} - (\ )_{i-1,j-1,k-1}\Big]$$

where $\Delta x = x_i - x_{i-1}$

$\Delta y = y_j - y_{j-1}$

$\Delta z = z_k - z_{k-1}$.

The second derivative in the z direction is obtained by defining new unknowns f, g say equal to $\partial u/dz$ and $\partial v/dz$. Then $\partial^2 u/\partial z^2 = \partial f/\partial z$ and $\partial^2 v/\partial z^2 = \partial g/\partial z$.

The box technique can only be used if v is positive and originally Cebeci, *et al.*[31] proposed a solution technique in which the solution marched in either the positive or negative y directions according to the sign of v . This technique poses programming problems (which appear to have been overcome in the method of Lindhout[26] to which we will shortly return) and so Cebeci, *et al.*[32] proposed two alternative techniques. The first of these, the zig-zag box scheme, combines the zig-zag and box methods when v is negative so that M becomes $x_{i-\frac{1}{2}}$, y_i, $z_{k-\frac{1}{2}}$ and now derivatives are written as

$$\left[\frac{\partial(\ )}{\partial x}\right]_M = \frac{1}{2\Delta x_{i-1}}\left[(\ )_{i,j,k} + (\ )_{i,j,k-1} - (\ )_{i-1,j,k} - (\ )_{i-1,j,k-1}\right]$$

$$\left[\frac{\partial(\ )}{\partial y}\right]_M = \frac{1}{4}\Big[\left\{(\ )_{i,j,k} + (\ )_{i,j,k-1} - (\ )_{i,j-1,k} - (\ )_{i,j-1,k-1}\right\}\Big/\Delta y_{j-1} + \left\{(\ )_{i-1,j+1,k} + (\ )_{i-1,j+1,k-1} - (\ )_{i-1,j,k} - (\ )_{i-1,j,k-1}\right\}\Big/\Delta y_{ji}\Big]$$

$$\left[\frac{\partial(\ )}{\partial z}\right]_M = \frac{1}{2\Delta z_{k-1}}\left[(\ )_{i,j,k} + (\ )_{i-1,j,k} - (\ )_{i,j,k-1} - (\ )_{i-1,j,k-1}\right]$$

and

$$(\ )_{i-\frac{1}{2},j,k-\frac{1}{2}} = \frac{1}{4}\left[(\ )_{i,j,k} + (\ )_{i,j,k-1} + (\ )_{i-1,j,k} + (\ )_{i-1,j,k-1}\right]$$

...... (31)

The use of these approximations does however re-introduce the stability restriction mentioned above together with the restriction on the variation of the y step sizes if second order accuracy is to be preserved. Cebeci, *et al.*'s other method in Ref 32 is the

characteristic difference scheme, Fig 4, in which the inertia terms in the momentum equations are written as

$$\frac{u}{h_1}\frac{\partial}{\partial x} + \frac{v}{h_2}\frac{\partial}{\partial y} = \frac{u}{h_1}\frac{\partial}{\partial \Gamma}\frac{\partial \Gamma}{\partial x} \tag{32}$$

where Γ is the direction of the streamline or characteristic through x, y, z. If $\Gamma_{i,j,k-\frac{1}{2}}$ cuts x_{i-1} constant, $z_{k-\frac{1}{2}}$ constant, at Q the equations centred now on $PQ/2$ where $P = x_i, x_j, z_{k-\frac{1}{2}}$ can be differenced as for example

$$\left[\frac{u}{h_1}\frac{\partial u}{\partial x} + \frac{v}{h_2}\frac{\partial u}{\partial y}\right] = \left[\frac{u}{h_1}\frac{\partial u}{\partial \Gamma}\frac{\partial \Gamma}{\partial x}\right] = \frac{1}{2}\left(\frac{u_P}{h_{1P}} + \frac{u_Q}{h_{1Q}}\right)\left(\frac{u_P - u_Q}{\Delta x}\right)$$

where
$$u_P = \tfrac{1}{2}\left[u_{i,j,k} + u_{i,j,k-1}\right]$$

$$u_Q = \tfrac{1}{2}\Big[\alpha\left(u_{i-1,j-1,k} + u_{i-1,j-1,k-1}\right) + (1-\alpha)\left(u_{i-1,j,k} + u_{i-1,j,k-1}\right)\Big] \qquad y_{j-1} < y_q < y_j$$

$$u_Q = \tfrac{1}{2}\Big[-\alpha\left(u_{i-1,j-1,k} + u_{i-1,j-1,k-1}\right) + (1-\alpha)\left(u_{i-1,j,k} + u_{i-1,j,k-1}\right)\Big] \qquad y_{j-1} < y_q < y_{j+1}$$

$$\alpha = (\Delta x v h_1)/(\Delta y u h_2) .$$

This scheme cannot however be simply applied to the continuity equation which is usually dealt with by a zig-zag scheme. For the characteristic scheme to remain second order accurate α must be evaluated at $PQ/2$ and u_q must be accurate to within $O(\Delta^3)$. This last condition can be satisfied by using quadratic interpolation through three points say y_{j-1}, y_j and y_{j+1}. Boyd[8] has compared these three Cebeci difference schemes for the Trondheim Trials[5] and NLR test cases and finds the characteristic scheme to be less robust and to require roughly three times the computation time. Boyd's preferred strategy is to use the box scheme unless any point on a z column requires the zig-zag in which case the zig-zag scheme is used for all that column. Cebeci, et al. found that the characteristic scheme allowed them to find solutions closer to separation than the other schemes but Boyd found that for the NLR test case these differences were very slight.

It was mentioned above that Cebeci originally adopted a marching scheme which depended upon the sign of v. Lindhout[26] uses just such a scheme and has devised an algorithm to define the optimal marching scheme. The advantage of this scheme is that the stability restriction on the x step is minimised, since the explicit operators tend only to be used as v changes sign and is therefore close to zero.

The end result of any of the implicit methods is a set of non-linear algebraic equations which are linearised by the use of iteration and then solved by standard matrix techniques. The need to use iteration makes comparisons of timings between the methods particularly difficult. Some comparative timings are given in Ref 7 but they are not reproduced here as Boyd[8] has found that the number of x steps in one example could be doubled with little or no increase in run times. This apparent anomaly occurs because the smaller step size produces a better initial guess at the next x station and hence fewer iterations are required. A similar mechanism may explain why Lemmerman[33] found that even the old explicit methods remained competitive.

The stability restriction may be removed altogether if the method is made implicit in both the y and z directions as is done by Nash and Scruggs[27], Patankar and Spalding[34] and Baker[35]. A simultaneous solution technique is then required for the whole y, z plane. Nash and Scruggs use an alternative direction implicit method, Patankar and Spalding their simple algorithm and Baker a finite-element approach. The use of such a fully implicit technique allows the inclusion of the diffusion in the y direction and this has been done by both Baker and Spalding. In addition if one includes the z-direction momentum equation the pressure may be allowed to vary with z although it becomes necessary to uncouple the pressure in the x momentum equation from that in the y and z momentum equations. For further details see Ref 34.

Comparisons between the results of the various difference methods may be found in Refs 5, 6, 7, 8 and 33 but as mentioned above all timings contained therein must be treated with extreme caution and certainly no clear 'best buy' has emerged to date.

No inverse finite-difference method has been published but Boyd[8] has suggested how a method using equations (1) to (4) or (12), (22) and (26) could be devised.

7 CONCLUSIONS

Calculation methods for three-dimensional turbulent boundary layers on practical shapes are now fairly widely available. Of the two types, integral and finite-difference, the integral methods retain their advantages of speed and robustness which make them particularly suitable for viscous-inviscid interaction calculations. The finite-difference methods are however not restricted by any velocity profile assumptions and allow more advanced turbulence models to be more readily incorporated. In addition they hold the promise that they rather than experiment may suggest new profile families for integral methods. The emergence of inverse methods both integral and finite-difference would appear to be imminent.

Note added in proof

No sooner were the above words committed to print than an inverse finite-difference method appeared. This method[36], by Formery and Delery, like the integral methods described in section 5, requires two boundary layer thicknesses to be specified. This may make its application to viscous inviscid interactive calculations difficult. It is not however restricted to infinite yawed wing flows and good agreement is shown with both infinite yawed wing and fully three-dimensional experiments. Difficulties produced by the characteristics coming from downstream are avoided in this method by using the FLARE approximation in which for example the term $\partial u/\partial x$ is set to zero if u is negative.

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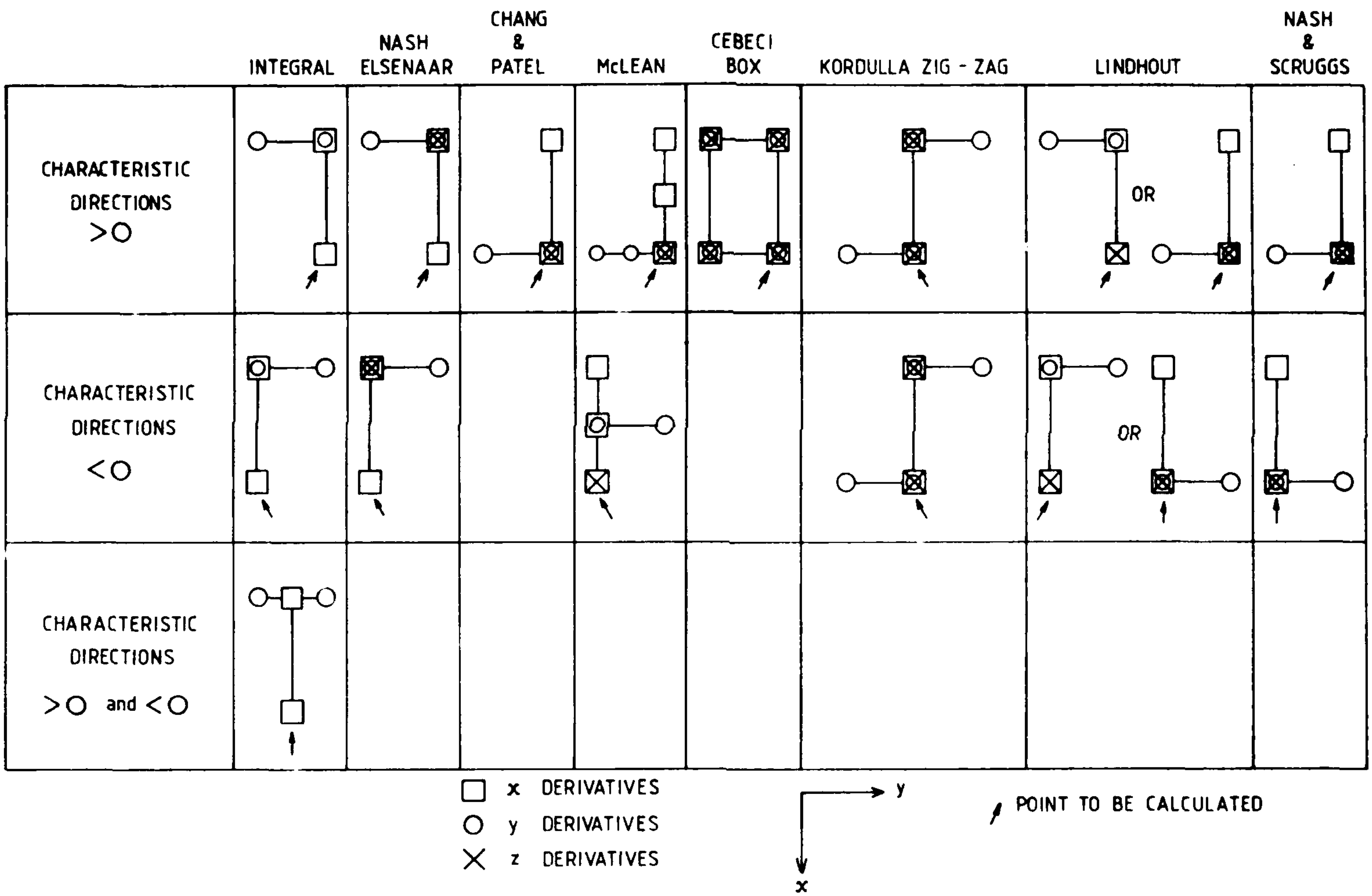
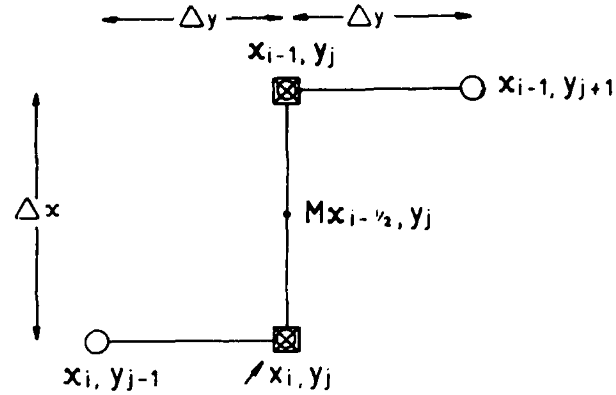


FIG.1 FINITE DIFFERENCE MOLECULES

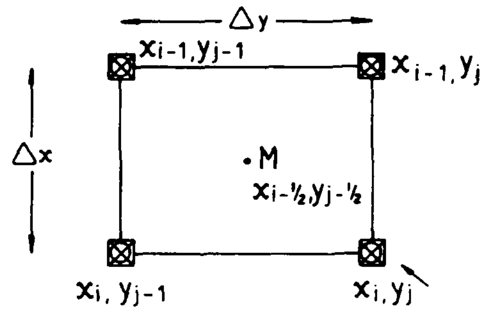


DIFFERENCE EQUATIONS CENTRED ON M

$$\left[\frac{\partial(\ )}{\partial x}\right]_M = \frac{1}{\Delta x}\left[(\quad)_{i,j} - (\quad)_{i-1,j}\right]$$

$$\left[\frac{\partial(\ )}{\partial y}\right]_M = \frac{1}{2\Delta y}\left[(\quad)_{i,j} - (\quad)_{i,j-1} + (\quad)_{i-1,j+1} - (\quad)_{i-1,j}\right]$$

FIG. 2 ZIG-ZAG DIFFERENCE SCHEME

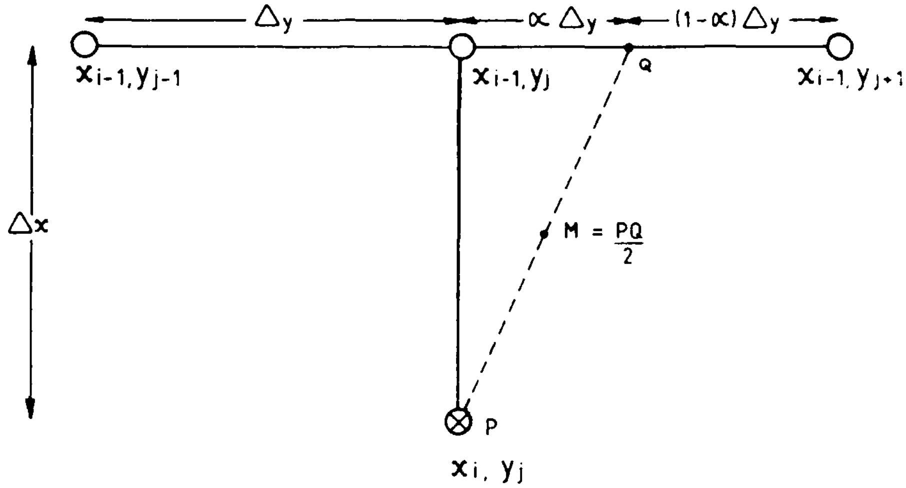


DIFFERENCE EQUATIONS CENTRED ON M

$$\left[\frac{\partial(\ )}{\partial x}\right]_M = \frac{1}{2\Delta x}\left[(\quad)_{i,j} + (\quad)_{i,j-1} - (\quad)_{i-1,j} - (\quad)_{i-1,j-1}\right]_{k-\frac{1}{2}}$$

$$\left[\frac{\partial(\ )}{\partial y}\right]_M = \frac{1}{2\Delta y}\left[(\quad)_{i,j} + (\quad)_{i-1,j} - (\quad)_{i,j-1} - (\quad)_{i-1,j-1}\right]_{k-\frac{1}{2}}$$

FIG. 3 BOX DIFFERENCE SCHEME



DIFFERENCE EQUATIONS CENTRED ON M

$$\left[\frac{u}{h_1}\frac{\partial u}{\partial x} + \frac{v}{h_2}\frac{\partial u}{\partial y}\right]_M = \left[\frac{u}{h_1}\frac{\partial u}{\partial \Gamma}\cdot\frac{d\Gamma}{dx}\right]_M = \frac{1}{2}\left(\frac{u_P}{h_{1P}} + \frac{u_Q}{h_{1Q}}\right)\cdot\left(\frac{u_P - u_Q}{\Delta x}\right)$$

$$u_Q = \frac{1}{2}\left[-\propto(u_{i-1,j+1,k} + u_{i-1,j+1,k-1}) + (1-\propto)(u_{i-1,j,k} + u_{i-1,j,k-1})\right]$$

$$\propto = (-\Delta x v h_1)/(\Delta y u h_2)$$

FIG.4 CHARACTERISTIC DIFFERENCE SCHEME

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17. Abstract

The governing equations for compressible three-dimensional turbulent boundary layers in a general coordinate system are given. Methods for the solution of these equations and their integral counterparts are described and compared. The available finite difference techniques are discussed in detail.

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